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Non-uniqueness of the Solution to Inverse Scattering Problem by Many Small Scatterers

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ABSTRACT
Inverse scattering problem by many small bodies is investigated. It is proved that the number of these bodies in a unit volume and the boundary impedance on the surface of these bodies cannot be recovered from the scattering data simultaneously. The basic assumption is a << d << λ, where a is the size of a small body, d is the distance between neighboring bodies, and λ is the wave length.

Keywords: Inverse Scattering, Many-body Scattering, Non-uniqueness Theorem.

MSC: 35P25; 81U10.

Introduction

Our result in this paper is based on the many-body wave scattering theory developed in [1], [2] and [4],[5]. Consider the wave scattering problem by many small bodies. Each of these bodies has the size a. This means that this body D_m can be put in a ball of radius a, centered at a point x_m, and a ball of radius $a/2$ centered at the point x_m belongs to the body, $1 \leq m \leq M$, where M is the number of small bodies. The basic assumption is:

$$a \ll d \ll \lambda, \tag{1}$$

where d is the minimal distance between neighboring bodies, and λ is the wave length.

In the cited author's monographs various boundary conditions are investigated.

In this paper we assume the impedance boundary condition on S_m, the surface of each of the small bodies:

$$u_N = \zeta_m u \quad \text{on } S_m, \quad \zeta_m = \frac{h(x_m)}{a^\kappa}, \; Im\zeta_m \leq 0. \tag{2}$$

Parameter $\kappa \in [0, 1)$ can be fixed arbitrarily by the experimenter, a is very small, the number M of small scatterers is large: $M = O(a^{\kappa-2})$ as $a \to 0$, N is the extrior unit normal on $S = \bigcup_{m=1}^M S_m$, the distribution of small bodies (scatterers) is given by the formula:

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0, \tag{3}$$

where $\Delta \subset D$ is an arbitrary open subset of D.

The scattering problem by many small scatterers with impedance boundary condition is:

$$(\Delta + k^2)u_M = 0 \text{ in } \mathbb{R}^3 \setminus \bigcup D_m, \tag{4}$$

$$\frac{\partial u_M}{\partial N} = \zeta_m u_M \quad \text{on } S_m, \tag{5}$$

where v, the scattered field, satisfies the radiation condition at infinity, and u_0 is the incident field, α is a unit vector and $k > 0$ is the wave number.

In p.41, wave scattering by many small scatterers, placed in an inhomogeneous medium is considered [4].

In this paper we assume for simplicity that the small scatterers are placed in a bounded domain D in a free space. It is proved in that the many-body scattering problem (4)-(5) has a solution, this solution is unique, and, as $a \to 0$ the solution tends uniformly to a limiting value $u(x)$, that satisfies the integral equation [4]:

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$$u(x) = u_0(x) - \int_D \frac{e^{ik|x-y|}}{|x-y|} h(y)N(y)u(y)dy. \quad (7)$$

This equation has a solution and the solution is unique (see [3], p. Its solution u solves the limiting scattering problem:

$$(\Delta + k^2 - 4\pi N(x)h(x))u = 0, \quad (8)$$

$$u_M = u_0 + v, \quad u_0 = e^{ik\alpha\cdot x}. \quad (9)$$

It is of interest to know that the theory developed in allows one to solve asymptotically (as $a \to 0$) exactly many-body scattering problem under the assumptions when the multiple scattering is essential, and to prove that the limiting field solves integral equation (6) [4]. It is proved in, p.264, that the scattering amplitude corresponding to the limiting scattering problem (7)-(8) determines uniquely the potential $p(x) := 4\pi N(x)h(x)$ [3].

Let us prove that the knowledge of $p(x)$ does not allow to determine $N(x)$ and $h(x)$ simultaneously.

Theorem 1. The scattering amplitude at a fixed $k > 0$ determines uniquely the potential $p(x) := 4\pi N(x)h(x)$, but cannot determine simultaneously $N(x)$ and the impedance boundary condition uniquely.

Proof. As was mentioned earlier, the scattering amplitude at a fixed $k > 0$ determines uniquely the potential $p(x) := 4\pi N(x)h(x)$.

One can fix $N(x) > 0$ arbitrarily, for example, to take $N(x) = c > 0$, where c is a constant. Then

$$h(x) = \frac{p(x)}{4\pi c}. \quad (10)$$

The $h(x)$ determines the boundary condition uniquely, see equation (2).
Theorem 1 is proved.

Conclusion

A non-uniqueness of the solution to an inverse scattering problem for many small bodies with an impedance boundary condition is proved under the assumption $a << d << \lambda$, for very small a. Here a is the size of a small body, d is the minimal distance between neighboring bodies, λ is the wavelength. One can determine $N(x)h(x)$ but not separately $N(x)$ and $h(x)$.

Disclosure Statement

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