

Soft Set Theory for Modeling Complex Systems

Ashlhan Sezgin^{1*}, and Zeynep Ay²

¹Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Türkiye

²Department of Mathematics, Graduate School of Natural and Applied Sciences, Amasya University, Amasya, Türkiye

***Corresponding author:**

Ashlhan Sezgin, Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Türkiye.

Abstract

Soft set theory constitutes a mathematically rigorous and structurally expressive formalism for modeling complex systems characterized by epistemic indeterminacy, vagueness, and parameter-dependent variability—features that pervade foundational problems across decision theory, engineering, economics, and information sciences. Central to this theoretical framework is a diverse repertoire of algebraic operations and binary product constructions, which collectively endow the universe of soft sets with a rich internal algebraic architecture capable of capturing intricate parametric dependencies. Within this context, we introduce and formally investigate a novel product of soft sets, designated as the “soft union–symmetric difference product”, defined over soft sets whose parameter domains are endowed with group-theoretic structure. Two key structural implications emerge from our investigation: (i) the incorporation of the soft union–symmetric difference product reinforces the internal operational coherence of soft set theory by embedding it into a formally consistent and axiomatically grounded algebraic environment; and (ii) the operation offers a foundational scaffold for the development of a generalized soft group theory, wherein group-parameterized soft sets simulate the axiomatic behavior of classical group structures under newly defined soft operations.

Keywords: Soft Set Theory, Complex Systems, Binary Product Constructions, Group Structures, Soft Operations.

Received: October 16, 2025;

Accepted: October 27, 2025;

Published: November 03, 2025

Introduction

A wide spectrum of advanced mathematical formalisms has been developed to model systems characterized by uncertainty, vagueness, and epistemic indeterminacy—phenomena ubiquitous in applied contexts ranging from engineering and economics to social sciences and medical diagnostics. Classical frameworks such as fuzzy set theory and probabilistic models, despite their expressive capabilities, are constrained by foundational assumptions that limit their generality. In particular, fuzzy set theory by Zadeh relies on subjectively assigned membership functions, while probabilistic models presuppose known distributions and repeated experiments—conditions that are frequently violated in systems governed by parameter-dependent or qualitative uncertainty [1]. To address these limitations, Molodtsov introduced soft set theory as a parameter-based framework designed to model uncertainty without reliance on probabilistic or membership-theoretic

axioms [2]. Its minimal axiomatic structure and inherent flexibility have rendered it particularly amenable to algebraic generalization, thus making it suitable for decision-making tool [3-23]. Since its inception, soft set theory has been equipped with an increasingly sophisticated suite of algebraic operations. Maji et al. laid the groundwork by introducing foundational operations such as union, intersection, and AND/OR-products, while Pei and Miao reinterpreted these operations within information-theoretic paradigms, extending the formalism to relational and multivalued domains [24, 25]. The algebraic landscape was further refined through the work of Ali et al., who defined restricted and extended variants, thereby enhancing the granularity of soft set operations [26]. Subsequent developments—by Yang, Feng et al., Jiang et al., Singh and Onyeozili, Zhu and Wen, and many others—have introduced a variety of binary product constructions and equality notions aimed at deepening the algebraic

Citation: Ashlhan Sezgin, and Zeynep Ay (2025) Soft Union-symmetric Difference Product of Groups. J All Phy Res Appl 1: 1-8.

structure of soft sets of particular relevance to algebraists is the evolution of soft equalities and soft subsethood relations [27-34].

Recent advances have substantially deepened the algebraic infrastructure of soft set theory through the introduction of a wide spectrum of innovative binary operations and structural refinements; each rigorously developed within formally defined algebraic settings. Notable contributions include the definition and axiomatic analysis of soft product operations over semigroups, groups, and rings, as well as the concept of subsets and equality relations tailored for uncertainty modeling. In particular, in collective efforts have established a robust, systematically extensible algebraic infrastructure that continues to propel the theoretical development of soft set theory [35-39]. Collectively, these investigations have established a mathematically rigorous and extensible foundation that enables the construction of sophisticated soft algebraic models capable of addressing complex problems in abstract algebra, logic, and decision theory under uncertainty.

The classical soft set introduced by Maji et al. was generalized by Pei and Miao and later formalized through soft congruences by Qin and Hong [25, 26, 40]. Jun and Yang extended this by defining J-soft equalities and associated distributive frameworks [41]. Liu et al. introduced L-soft subsets and L-equalities, highlighting cases where classical distributive identities fail [42]. These developments culminated in categorical frameworks advanced by Feng and Yongming, who examined associativity, commutativity, and distributivity under generalized equality schemes and demonstrated that certain classes of quotient soft algebras form commutative semigroups [43]. More recent generalizations—such as g-soft, gf-soft, and T-soft equalities—have been framed within lattice-theoretic and congruence-based settings by Abbas et al., Alshami, and Alshami and El-Shafei, reflecting a methodological shift toward structured algebraic foundations [44-47]. Simultaneously, Çağman and Enginoğlu reformulated the axiomatic basis of soft set operations, resolving critical inconsistencies and establishing a stable algebraic framework [48]. The soft intersection–union product has been adapted to rings by Sezer, yielding well-formed structures such as soft union rings [49]. Conversely, the soft union–intersection product has been explored within group-theoretic by Kaygısız and semigroup-theoretic by Sezer et al., contexts, with structural properties often contingent on the algebraic behavior of identity and inverse elements in the parameter domain [50, 51].

Against this backdrop, the present study introduces a novel product—the soft union–symmetric difference product—defined over soft sets whose parameter spaces are structured as groups. This operation is rigorously formalized and subjected to detailed algebraic analysis, with particular attention to closure, associativity, commutativity, idempotency, the compatibility of the product with generalized soft subsethood and equality relations, along with its behavior with respect to both null and absolute soft sets. In addition, the interplay between the proposed product and previously established soft product operations is thoroughly investigated within the framework of soft subset classifications, offering refined insights into their relative expressive capacities and mutual structural compatibilities. Our results indicate that the proposed product induces a well-defined product on the collection

of soft sets, thereby enabling a natural extension of classical group-theoretic principles to the soft context. In doing so, this work lays the conceptual foundation for a soft group theory, wherein algebraic systems defined over group-parameterized soft sets obey suitably adapted axioms and binary operations. The remainder of this paper is organized as follows. Section 2 recalls necessary preliminaries and foundational definitions related to soft sets and soft equalities. Section 3 introduces the soft union–symmetric difference product and develops its algebraic theory, including structural theorems and illustrative examples. Section 4 summarizes the principal findings and outlines directions for future work, particularly with respect to the development of soft algebraic systems and their applications in abstract algebra, formal semantics, and uncertainty modeling.

Preliminaries

This section presents a rigorous and methodical re-evaluation of the foundational definitions and algebraic underpinnings that serve as the formal substrate for the theoretical constructs elaborated in the subsequent discourse. While the original conception of soft set theory was introduced by Molodtsov as a parameterized generalization for modeling uncertainty, its formal definitional schema and operational calculus were substantially restructured in the influential reformulation by Çağman and Enginoğlu [2, 48]. Their axiomatic revision endowed the theory with heightened structural coherence and broadened its applicability across diverse algebraic and decision-theoretic settings. The present investigation adopts this refined formalism as the axiomatic foundation upon which all further constructions are based. Accordingly, every algebraic development, operational specification, and theoretical generalization in the forthcoming sections is rigorously articulated within this enhanced framework, ensuring both internal consistency and formal adherence to contemporary standards in soft algebraic systems.

Definition 2.1. Let E be a parameter set, U be a universal set, $P(U)$ be the power set of U , and $H \subseteq E$ [48]. Then, the soft set over U is a function such that $f_H: E \rightarrow P(U)$, where for all $w \in H$, $f_H(w) = \emptyset$. That is,

$$f_H = \{(w, f_H(w)) : w \in E\}$$

From now on, the soft set over U is abbreviated by SS .

Definition 2.2. Let f_H be an SS [48]. If $f_H(w) = \emptyset$ for all $w \in E$, then f_H is called a null SS and indicated by $\emptyset_E$, and if $f_H(w) = U$, for all $w \in E$, then f_H is called an absolute SS and indicated by U_E .

Definition 2.3. Let f_H and g_K be two SS s [48]. If $f_H(w) \subseteq g_K(w)$, for all $w \in E$, then f_H is a soft subset of g_K and indicated by $f_H \subseteq g_K$. If $f_H(w) = g_K(w)$, for all $w \in E$, then f_H is called soft equal to g_K and denoted by $f_H = g_K$.

Definition 2.4. Let f_H and g_K be two SS s [48]. Then, the difference of f_H and g_K is the SS $f_H \tilde{\setminus} g_K$, where $(f_H \tilde{\setminus} g_K)(w) = f_H(w) \setminus g_K(w)$, for all $w \in E$.

Definition 2.5. Let f_H be an SS [48]. Then, the complement of f_H denoted by f_H^c , is defined by the soft set $f_H^c: E \rightarrow P(U)$ such that $f_H^c(e) = U \setminus f_H(e) = (f_H(e))'$, for all $e \in E$.

Definition 2.6. Let f_K and g_N be two SSs [52]. Then, f_K is called a soft S-subset of g_N , denoted by $f_K \subseteq_S g_N$, if for all $w \in E$, $f_K(w) = M$ and $g_N(w) = D$, where M and D are two fixed sets and $M \subseteq D$. Moreover, two SSs f_K and g_N are said to be soft S-equal, denoted by $f_K =_S g_N$, if $f_K \subseteq_S g_N$ and $g_N \subseteq_S f_K$.

It is obvious that if $f_K =_S g_N$, then f_K and g_N are the same constant functions, that is, for all $w \in E$, $f_K(w) = g_N(w) = M$, where M is a fixed set.

Definition 2.7. Let f_K and g_N be two SSs [52]. Then, f_K is called a soft A-subset of g_N , denoted by $f_K \subseteq_A g_N$, if, for each $a, b \in E$, $f_K(a) \subseteq g_N(b)$.

Definition 2.8. Let f_K and g_N be two SSs [52]. Then, f_K is called a soft S-complement of g_N , denoted by $f_K =_S (g_N)^c$, if, for all $w \in E$, $f_K(w) = M$ and $g_N(w) = D$, where M and D are two fixed sets and $M = D^c$. Here, $D^c = U \setminus D$.

From now on, let G be a group, and $S_G(U)$ denotes the collection of all SSs over U , whose parameter sets are G ; that is, each element of $S_G(U)$ is an SS parameterized by G .

From now on, the symmetric difference of the family $\mathfrak{B} = \{C_i : i \in I\}$ such that I is an index set, is denoted by

$$\Delta \mathfrak{B} = \Delta_{i \in I} C_i = C_1 \Delta C_2 \Delta \dots \Delta C_n$$

for all $x \in G$.

Definition 2.9. Let f_G and g_G be two SSs [52]. Then, the soft symmetric difference-product $f_G \otimes_{s/d} g_G$ is defined by

$$(f_G \otimes_{s/d} g_G)(x) = \Delta_{y=yz} (f_G(y) \setminus g_G(z)), (y, z \in G)$$

for all $x \in G$.

Definition 2.10. Let f_G and g_G be two SSs [53]. Then, the intersection-difference product $f_G \otimes_{i/d} g_G$ is defined by

$$(f_G \otimes_{i/d} g_G)(x) = \bigcap_{y=yz} (f_G(y) \cap g_G(z)), (y, z \in G)$$

for all $x \in G$.

For additional information on SSs and Picture SS, we refer to [54-106].

Soft Union-symmetric Difference Product

In this section, we introduce a novel product on SSs, termed the soft union-symmetric difference product, defined over parameter spaces endowed with group structures. A detailed algebraic investigation is conducted to rigorously characterize the foundational structural properties of this product. Particular emphasis is placed on analyzing its interaction with various generalized notions of soft equality and on the stratification of soft subsets under multiple inclusion criteria. To bridge the gap between abstract formalism and concrete intuition, the theoretical exposition is supplemented with carefully selected examples that illustrate the operational behavior and highlight key algebraic phenomena associated with the product. Furthermore, the relationships between the proposed product and several existing soft products together with null and absolute SS are examined

with respect to soft subethood, thereby clarifying its algebraic compatibility within the broader operational framework of SS theory. This analysis underscores the structural coherence of the product and its potential to serve as a foundational component in the development of more comprehensive soft algebraic systems as possible to the text they refer to and aligned center.

Definition 3.1. Let f_G and g_G be two SSs. Then, the soft union-symmetric difference product $f_G \otimes_{u/s} g_G$ is defined by

$$(f_G \otimes_{u/s} g_G)(x) = \bigcup_{y=yz} (x=yz)(f_G(y) \Delta g_G(z)), y, z \in G$$

for all $x \in G$.

Note here that since G is a group, there always exist $y, z \in G$ such that $x = yz$, for all $x \in G$. Let the order of the group G be n , that is, $|G| = n$. Then, it is obvious that there exist n different combinations of writing styles for each $x \in G$ such that $x = yz$, where $y, z \in G$.

Note 3.2. The soft union-symmetric difference product is well-defined in $S_G(U)$. In fact, let $f_G, g_G, p_G, n_G \in S_G(U)$ such that $(f_G, g_G) = (p_G, n_G)$. Then, $f_G = p_G$ and $g_G = n_G$, implying that $f_G(x) = p_G(x)$ and $g_G(x) = n_G(x)$, for all $x \in G$. Thereby, for all $x \in G$,

$$\begin{aligned} (f_G \otimes_{u/s} g_G)(x) &= \bigcup_{y=yz} (f_G(y) \Delta g_G(z)) \\ &= \bigcup_{y=yz} (p_G(y) \Delta n_G(z)) \\ &= (p_G \otimes_{u/s} n_G)(x) \end{aligned}$$

Hence, $f_G \otimes_{u/s} g_G = p_G \otimes_{u/s} n_G$.

Example 3.3. Consider the group $G = \{u, w\}$ with the following operation:

	u	w
u	u	w
w	w	u

Let f_G and g_G be two SSs over $U = D_2 = \{<x, y> : x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:

$$f_G = \{(u, \{x, yx\}), (w, \{x, y\})\} \text{ and } g_G = \{(u, \{e, y, yx\}), (w, \{x, y\})\}$$

Since $u = uu = ww$, $(f_G \otimes_{u/s} g_G)(u) = (f_G(u) \Delta g_G(u)) \cup (f_G(w) \Delta g_G(w)) = \{e, x, y\}$, and since $w = uw = wu$, $(f_G \otimes_{u/s} g_G)(w) = (f_G(u) \Delta g_G(w)) \cup (f_G(w) \Delta g_G(u)) = \{e, x, y, yx\}$ is obtained. Hence, $f_G \otimes_{u/s} g_G = \{(u, \{e, x, y\}), (w, \{e, x, y, yx\})\}$

Proposition 3.4. The set $S_G(U)$ is closed under the soft union-symmetric difference product. That is, if f_G and g_G are two SSs, then so is $f_G \otimes_{u/s} g_G$.

PROOF. It is obvious that the soft union-symmetric difference product is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the soft union-symmetric difference product.

Proposition 3.5. The soft union-symmetric difference product is not associative in $S_G(U)$.

PROOF. Consider the group G and the SSs f_G and g_G in Example 3.3, and let $h_G = \{(u, \{y, yx\}), (w, \{e, x, y\})\}$ be an SS over $U = \{e, x, y, yx\}$.

Since $f_G \otimes_{w/s} g_G = \{(u, \{e, x, y\}), (w, \{e, x, y, yx\})\}$, then $(f_G \otimes_{w/s} g_G) \otimes_{w/s} h_G = \{(u, \{e, x, yx\}), (w, \{e, x\})\}$

Moreover, since $g_G \otimes_{w/s} h_G = \{(u, \{e\}), (w, \{x, yx\})\}$, then

$$f_G \otimes_{w/s} (g_G \otimes_{w/s} h_G) = \{(u, \{e, x, y, yx\}), (w, \{e, x, y\})\}$$

Thereby, $(f_G \otimes_{w/s} g_G) \otimes_{w/s} h_G \neq f_G \otimes_{w/s} (g_G \otimes_{w/s} h_G)$.

Proposition 3.6. The soft union-symmetric difference product is not commutative in $S_G(U)$. However, if G is an abelian group, then the soft union-symmetric difference product is commutative in $S_G(U)$.

PROOF. Let f_G and g_G be two SSs and G be an abelian group. Then, for all $x \notin G$,

$$\begin{aligned} (f_G \otimes_{w/s} g_G)(x) &= \bigcup_{x=yz} (f_G(y) \Delta g_G(z)) \\ &= \bigcup_{x=yz} (g_G(z) \Delta f_G(y)) \\ &= (g_G \otimes_{w/s} f_G)(x) \end{aligned}$$

implying that $f_G \otimes_{w/s} g_G = g_G \otimes_{w/s} f_G$.

Proposition 3.7. The soft union-symmetric difference product is not idempotent in $S_G(U)$.

PROOF. Consider the SS f_G in Example 3.3. Then,

$$f_G \otimes_{w/s} f_G = \{(u, \emptyset), (w, \{y, yx\})\}$$

implying that $f_G \otimes_{w/s} f_G \neq f_G$.

Proposition 3.8. Let f_G be a constant SS. Then, $f_G \otimes_{w/s} f_G = \emptyset_G$.

PROOF. Let f_G be a constant SS such that, for all $x \notin G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \notin G$,

$$(f_G \otimes_{w/s} f_G)(x) = \bigcup_{x=yz} (f_G(y) \Delta f_G(z)) = \emptyset_G(x)$$

Thereby, $f_G \otimes_{w/s} f_G = \emptyset_G$.

Remark 3.9. Let $S_G^*(U)$ be the collection of all constant SSs. Then, the soft union-symmetric difference product is not idempotent in $S_G^*(U)$ either.

Proposition 3.10. Let f_G be a constant SS. Then, $f_G \otimes_{w/s} \emptyset_G = \emptyset_G \otimes_{w/s} f_G = f_G$.

PROOF. Let f_G be a constant SS such that, for all $x \notin G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \notin G$,

$$(f_G \otimes_{w/s} \emptyset_G)(x) = \bigcup_{x=yz} (f_G(y) \Delta \emptyset_G(z)) = \bigcup_{x=yz} (f_G(y) \Delta \emptyset) = f_G(x)$$

Thereby, $f_G \otimes_{w/s} \emptyset_G = f_G$. Similarly, for all $x \notin G$,

$$(\emptyset_G \otimes_{w/s} f_G)(x) = \bigcup_{x=yz} (\emptyset_G(y) \Delta f_G(z)) = \bigcup_{x=yz} (\emptyset \Delta f_G(z)) = f_G(x)$$

Thereby, $\emptyset_G \otimes_{w/s} f_G = f_G$.

Remark 3.11. $\emptyset_G$ is the identity element of the soft union-symmetric difference product in $S_G^*(U)$. Besides, the inverse of each element is itself in $S_G^*(U)$ with respect to the soft union-symmetric difference product by Proposition 3.8.

Proposition 3.12. Let f_G be a constant SS. Then, $f_G \otimes_{w/s} U_G = U_G \otimes_{w/s} f_G = f_G^c$.

PROOF. Let f_G be a constant SS such that, for all $x \notin G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \notin G$,

$$(f_G \otimes_{w/s} U_G)(x) = \bigcup_{x=yz} (f_G(y) \Delta U_G(z)) = \bigcup_{x=yz} (f_G(y) \Delta U) = f_G^c(x)$$

Thereby, $f_G \otimes_{w/s} U_G = f_G^c$. Similarly, for all $x \notin G$,

$$(U_G \otimes_{w/s} f_G)(x) = \bigcup_{x=yz} (U_G(y) \Delta f_G(z)) = \bigcup_{x=yz} (U \Delta f_G(z)) = f_G^c(x)$$

Thereby, $U_G \otimes_{w/s} f_G = f_G^c$.

Proposition 3.13. Let f_G be a constant SS. Then, $f_G \otimes_{w/s} f_G^c = f_G^c \otimes_{w/s} f_G = U_G$.

PROOF. Let f_G be a constant SS such that, for all $x \notin G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \notin G$,

$$(f_G \otimes_{w/s} f_G^c)(x) = \bigcup_{x=yz} (f_G(y) \Delta f_G^c(z)) = U = U_G(x)$$

Thereby, $f_G \otimes_{w/s} f_G^c = U_G$. Similarly, for all $x \notin G$,

$$(f_G^c \otimes_{w/s} f_G)(x) = \bigcup_{x=yz} (f_G^c(y) \Delta f_G(z)) = U = U_G(x)$$

Thus, $f_G^c \otimes_{w/s} f_G = U_G$.

Proposition 3.14. Let f_G and g_G be two SSs. Then, $f_G \otimes_{w/s} g_G = \emptyset_G$ if and only if $f_G =_s g_G$.

PROOF. Let f_G and g_G be two SSs. Suppose that $f_G =_s g_G$. Hence, for all $u \notin G$, $f_G(u) = M$ and $g_G(u) = D$, where M and D are two fixed sets and $M = D$. Thus, for all $x \notin G$,

$$(f_G \otimes_{w/s} g_G)(x) = \bigcup_{x=yz} (f_G(y) \Delta g_G(z)) = \emptyset = \emptyset_G(x)$$

Thereby, $f_G \otimes_{w/s} g_G = \emptyset_G$.

Conversely, suppose that $f_G \otimes_{w/s} g_G = \emptyset_G$. Then, $(f_G \otimes_{w/s} g_G)(x) = \emptyset_G(x) = \emptyset$, for all $x \notin G$. Thus, for all $x \notin G$,

$$\emptyset_G(x) = \emptyset = (f_G \otimes_{w/s} g_G)(x) = \bigcup_{x=yz} (f_G(y) \Delta g_G(z))$$

This implies that $f_G(x) \Delta g_G(y) = \emptyset$, for all $x, y \notin G$. Thus, $f_G(x) = g_G(y)$ for all $x, y \notin G$. Thereby, $f_G =_s g_G$.

Proposition 3.15. Let f_G and g_G be two SSs. If $f_G =_s (g_G)^c$, then $f_G \otimes_{w/s} g_G = U_G$.

PROOF. Let f_G and g_G be two SSs. Suppose that $f_G =_s (g_G)^c$. Hence, for all $u \notin G$, $f_G(u) = M$ and $g_G(u) = D$, where M and D are two fixed sets and $M = D'$. Thus, for all $x \notin G$,

$$(f_G \otimes_{w/s} g_G)(x) = \bigcup_{x=yz} (f_G(y) \Delta g_G(z)) = U = U_G(x)$$

Thereby, $f_G \otimes_{w/s} g_G = U_G$.

Proposition 3.16. Let f_G and g_G be two SSs. Then, $(f_G \otimes_{w/s} g_G)^c = f_G \otimes_{w/s} g_G$.

PROOF. Let f_G and g_G be two SSs. Then, for all $x \notin G$,

$$\begin{aligned} (f_G \otimes_{w/s} g_G)(x) &= \bigcup_{x=yz} (f_G(y) \Delta g_G(z)) \\ &= \bigcap_{x=yz} (f_G(y) \Delta g_G(z))' \end{aligned}$$

$$= \bigcap_{x \neq y} (f_G(y) \cup g_G(z)) \\ = (f_G \otimes_{i/s} g_G)^c(x)$$

Thus, $(f_G \otimes_{u/s} g_G)^c = f_G \otimes_{i/s} g_G$. For more on the symmetric difference complement ($\cup$) operation, we refer to [107].

Proposition 3.17. Let f_G and g_G be two SSs, where G is an abelian group. If $f_G \subseteq_A g_G$, then $f_G \otimes_{i/s} g_G = g_G \otimes_{u/d} f_G$.

PROOF. Let f_G and g_G be two SSs such that $f_G \subseteq_A g_G$. Then, for each $x, y \notin G$, $f_G(x) \subseteq g_G(y)$. Thus, for all $x \notin G$,

$$(f_G \otimes_{u/s} g_G)(x) = \bigcup_{y \neq z} (f_G(y) \Delta g_G(z)) = \bigcup_{y \neq z} (g_G(z) \setminus f_G(y)) = (g_G \otimes_{u/s} f_G)(x)$$

Thus, $(f_G \otimes_{u/s} g_G = g_G \otimes_{u/s} f_G$.

Remark 3.18. Let f_G and g_G be two SSs. If $f_G \subseteq_s g_G$, then $f_G \otimes_{u/s} g_G = g_G \otimes_{u/s} f_G$.

Proposition 3.19. Let f_G and g_G be two SSs. Then, $f_G \otimes_{s/d} g_G \subseteq f_G \otimes_{u/s} g_G$.

PROOF. Let f_G and g_G be two SSs. Then, for all $x \notin G$,

$$(f_G \otimes_{s/d} g_G)(x) = \bigtriangleup_{x \neq y \neq z} (f_G(y) \setminus g_G(z)) \\ \subseteq \bigcup_{y \neq z} (f_G(y) \Delta g_G(z)) = (f_G \otimes_{u/s} g_G)(x)$$

There by, $f_G \otimes_{s/d} g_G \subseteq f_G \otimes_{u/s} g_G$.

Proposition 3.20. Let f_G and g_G be two SSs. If one of the following assertions is satisfied, then $f_G \otimes_{s/d} g_G = f_G \otimes_{u/s} g_G$.

i. $g_G \subseteq_s f_G$ and $|G| = n$, where n is a positive odd integer

ii. $f_G = U_G$ and $|G| = n$, where n is a positive odd integer

iii. $f_G =_s g_G$

PROOF. Let f_G and g_G be two SSs.

i. Suppose that $g_G \subseteq_s f_G$. Hence, for all $u \notin G$, $f_G(u) = M$ and $g_G(u) = D$, where M and D are two fixed sets and $D \subseteq M$. Thus, for all $x \notin G$,

$$(f_G \otimes_{s/d} g_G)(x) = \bigtriangleup_{x \neq y \neq z} (f_G(y) \setminus g_G(z)) \\ = \bigcup_{y \neq z} (f_G(y) \Delta g_G(z)) \\ = (f_G \otimes_{u/s} g_G)(x)$$

There by, $f_G \otimes_{s/d} g_G = f_G \otimes_{u/s} g_G$.

ii. Suppose that $f_G = U_G$ and $|G| = n$, where n is a positive odd integer. Hence, for all $u \notin G$, $f_G(u) = U_G(u) = U$. Thus, for all $x \notin G$,

$$(f_G \otimes_{s/d} g_G)(x) = \bigtriangleup_{x \neq y \neq z} (f_G(y) \setminus g_G(z)) \\ = \bigcup_{y \neq z} (f_G(y) \Delta g_G(z)) \\ = (f_G \otimes_{u/s} g_G)(x)$$

Thereby, $f_G \otimes_{s/d} g_G = f_G \otimes_{u/s} g_G$.

iii. Suppose that $f_G =_s g_G$. Hence, for all $u \notin G$, $f_G(u) = M$ and $g_G(u) = D$, where M and D are two fixed sets and $M = D$. Thus, for all $x \notin G$,

$$(f_G \otimes_{s/d} g_G)(x) = \bigtriangleup_{x \neq y \neq z} (f_G(y) \setminus g_G(z)) \\ = \bigcup_{y \neq z} (f_G(y) \Delta g_G(z)) \\ = (f_G \otimes_{u/s} g_G)(x)$$

Thereby, $f_G \otimes_{s/d} g_G = f_G \otimes_{u/s} g_G$.

Proposition 3.21. Let f_G and g_G be two SSs. Then, $f_G \otimes_{i/d} g_G \subseteq f_G \otimes_{u/s} g_G$.

PROOF. Let f_G and g_G be two SSs. Then, for all $x \notin G$,

$$(f_G \otimes_{i/d} g_G)(x) = \bigcap_{x \neq y \neq z} (f_G(y) \setminus g_G(z)) \\ \subseteq \bigcup_{x \neq y \neq z} (f_G(y) \Delta g_G(z))$$

$$= (f_G \otimes_{u/s} g_G)(x)$$

Thereby, $f_G \otimes_{i/d} g_G \subseteq f_G \otimes_{u/s} g_G$.

Proposition 3.22. Let f_G and g_G be two SSs. If $g_G \subseteq_s f_G$ then, $f_G \otimes_{i/d} g_G = f_G \otimes_{u/s} g_G$.

PROOF. Let f_G and g_G be two SSs. Suppose that $g_G \subseteq_s f_G$. Hence, for all $u \notin G$, $f_G(u) = M$ and $g_G(u) = D$, where M and D are two fixed sets and $D \subseteq M$. Thus, for all $x \notin G$,

$$(f_G \otimes_{i/d} g_G)(x) = \bigcap_{x \neq y \neq z} (f_G(y) \setminus g_G(z)) \\ = \bigcup_{x \neq y \neq z} (f_G(y) \Delta g_G(z)) \\ = (f_G \otimes_{u/s} g_G)(x)$$

Thereby, $f_G \otimes_{i/d} g_G = f_G \otimes_{u/s} g_G$.

Conclusion

This study begins by formally introducing a novel product of soft sets, termed the soft union–symmetric difference product, defined over parameter spaces endowed with a group-theoretic structure. Building upon this foundational construct, we undertake a rigorous algebraic analysis of the product, with particular emphasis on its behavior under various taxonomies of soft subsethood and its alignment with generalized notions of soft equality. The systematic formulation and examination of such operations within an axiomatized framework align with the core methodology of abstract algebra, where structural classification is predicated on the verification of key properties—such as closure, associativity, commutativity, idempotency, and the presence (or absence) of identity, inverse, and absorbing elements. In addition, we explore the product's interaction with existing soft binary operations together with its relation with null and absolute soft set, with a focus on its algebraic behavior relative to soft subset and equality structures. The structural insights gained from this analysis not only reinforce the internal logical coherence of the proposed framework but also demonstrate the product's potential to generalize classical algebraic constructs, thereby extending the expressive and applicative capacities of soft algebraic systems. The theoretical foundation articulated herein addresses notable gaps in the existing literature and paves the way for the development of soft group theory—a new line of inquiry grounded in rigorously defined soft operations over group-structured parameter domains. Future research directions may involve the synthesis of additional algebraic operations in soft environments and the refinement of generalized equality frameworks, contributing to the ongoing expansion of soft set theory's role in abstract algebra, computational modeling, and uncertainty-aware decision analysis.

Funding Support: No funding.

Ethical Statement: This study does not contain any studies with human or animal subjects performed by any of the authors.

Conflicts of Interest: The authors declare that they have no conflicts of interest to this work.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Author contributions: Conceptualization, AS and ZA; methodology, AS; software, ZA; validation, AS and ZA; formal analysis, ZA; investigation, AS and ZA; resources, AS; data curation, ZA; writing—original draft preparation, ZA; writing—review and editing, ZA and AS; visualization, ZA; supervision, AS. All authors have read and agreed to the published version of the manuscript.

References

1. Zadeh LA (1965) Fuzzy sets. *Information and Control* 8: 338-353.
2. Molodtsov D (1999) Soft set theory-First results. *Computers and Mathematics with Applications* 37: 19-31.
3. Maji PK, Roy AR, Biswas R (2002) An application of soft sets in a decision making problem. *Computers and Mathematics with Applications* 44: 1077-1083.
4. Chen DG, Tsang ECC, Yeung DS (2003) Some notes on the parameterization reduction of soft sets. *Proceedings of the 2003 International Conference on Machine Learning and Cybernetics* 1442-1445.
5. Chen DG, Tsang ECC, Wang X (2005) The parametrization reduction of soft sets and its applications. *Computers and Mathematics with Applications* 49: 757-763.
6. Mushrif MM, Sengupta S, Ray AK (2006) Texture classification using a novel, soft-set theory based classification algorithm. In: Narayanan PJ, Nayar SK, Shum HT, eds. *Computer Vision - ACCV 2006. Lecture Notes in Computer Science*. Berlin, Heidelberg: Springer; 3851.
7. Xiao Z, Chen L, Zhong B, Ye S (2005) Recognition for soft information based on the theory of soft sets. *IEEE Proceedings of International Conference on Services Systems and Services Management* 1104-1106.
8. Herawan MT, Deris MM (2009) A direct proof of every rough set is a soft set. *Third Asia International Conference on Modelling & Simulation* 119-124. Bundang, Indonesia.
9. Herawan MT, Deris MM (2010) Soft decision making for patients suspected influenza. In: Taniar D, Gervasi O, Murgante B, Pardede E, Apduhan BO, eds. *Computational Science and Its Applications - ICCSA 2010. Lecture Notes in Computer Science*. Berlin, Heidelberg: Springer; 6018.
10. Herawan T (2005) Soft set-based decision making for patients suspected influenza-like illness. *International Journal of Modern Physics: Conference Series* 1: 1-5.
11. Çağman N, Enginoğlu S (2010) Soft matrix theory and its decision making. *Computers and Mathematics with Applications* 59: 3308-3314.
12. Gong X, Xiao Z, Zhang X (2010) The bijective soft set with its operations. *Computers and Mathematics with Applications* 60: 2270-2278.
13. Xiao Z, Gong K, Xia S, Zou Y (2010) Exclusive disjunctive soft sets. *Computers and Mathematics with Applications* 59: 2128-2137.
14. Feng F, Li Y, Çağman N (2012) Generalized uni-int decision making schemes based on choice value soft sets. *European Journal of Operational Research* 220: 162-170.
15. Feng Q, Zhou Y (2014) Soft discernibility matrix and its applications in decision making. *Applied Soft Computing* 24: 749-756.
16. Kharal A (2014) Soft approximations and uni-int decision making. *The Scientific World Journal* 4: 327408.
17. Dauda MK, Mamat M, Waziri MY (2015) An application of soft set in decision making. *Jurnal Teknologi* 77: 119-122.
18. Inthumathi V, Chitra V, Jayasree S (2017) The role of operators on soft set in decision making problems. *International Journal of Computational and Applied Mathematics* 12: 899-910.
19. Atagün AO, Kamacı H, Oktay O (2018) Reduced soft matrices and generalized products with applications in decision making. *Neural Computing and Applications* 29: 445-456.
20. Kamacı H, Saltık K, Akız HF, Atagün AO (2018) Cardinality inverse soft matrix theory and its applications in multicriteria group decision making. *Journal of Intelligent & Fuzzy Systems* 34: 2031-2049.
21. Yang JL, Yao YY (2020) Semantics of soft sets and three-way decision with soft sets. *Knowledge-Based Systems* 194: 105538.
22. Petchimuthu S, Garg H, Kamacı H, Atagün AO (2020) The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM. *Computational and Applied Mathematics* 39: 1-32.
23. Zorlutuna İ (2021) Soft set-valued mappings and their application in decision making problems. *Filomat* 35: 1725-1733.
24. Maji PK, Biswas R, Roy AR (2003) Soft set theory. *Computers and Mathematics with Applications* 45: 555-562.
25. Pei D, Miao D (2005) From soft sets to information systems. In: *Proceedings of the IEEE International Conference on Granular Computing 2005*: 617-621.
26. Ali MI, Feng F, Liu X, Min WK, Shabir M (2009) On some new operations in soft set theory. *Computers & Mathematics with Applications* 57: 1547-1553.
27. Yang CF (2008) Soft set theory. [(2003) *Computers & Mathematics with Applications* 45: 555-562]. *Computers & Mathematics with Applications* 56: 1899-1900.
28. Feng F, Li C, Davvaz B, Irfan Ali M (2010) Soft sets combined with fuzzy sets and rough sets: a tentative approach. *Soft Computing* 14: 899-911.
29. Jiang Y, Tang Y, Chen Q, Wang J, Tang S (2010) Extending soft sets with description logics. *Computers & Mathematics with Applications* 59: 2087-2096.
30. Singh D, Onyeozili IA (2012) Notes on soft matrices operations. *ARPN Journal of Science and Technology* 2: 861-869.
31. Singh D, Onyeozili IA (2012) On some new properties of soft set operations. *International Journal of Computer Applications* 59: 39-44.
32. Singh D, Onyeozili IA (2012) Some results on distributive and absorption properties on soft operations. *IOSR Journal of Mathematics* 4: 18-30.
33. Singh D, Onyeozili IA (2012) Some conceptual misunderstandings of the fundamentals of soft set theory. *ARPN Journal of Systems and Software* 2: 251-254.
34. Zhu P, Wen Q (2013) Operations on Soft Sets Revisited. *Journal of Applied Mathematics* 2013: 1-7.

35. Eren ÖF, Çalışıcı H (2019) On some operations of soft sets. In: Proceedings of the Fourth International Conference on Computational Mathematics and Engineering Sciences.
36. Stojanović N (2021) A new operation on soft sets: extended symmetric difference of soft sets. *Military Technical Gazette* 69: 779-791.
37. Sezgin A, Çağman N, Atagün AO, Aybek FN (2023) Complemental binary operations of sets and their application to group theory. *Matrix Science Mathematic* 7: 114-121.
38. Sezgin A, Çalışıcı H (2024) A comprehensive study on soft binary piecewise difference operation. *Eskişehir Technical University Journal of Science and Technology B – Theoretical Sciences* 12: 32–54.
39. Sezgin A, Dagtoros K (2023) Complementary soft binary piecewise symmetric difference operation: A novel soft set operation. *Scientific Journal of Mehmet Akif Ersoy University* 6: 31–45.
40. Qin K, Hong Z (2010) On soft equality. *Journal of Computational and Applied Mathematics* 234: 1347-1355.
41. Jun YB, Yang X (2009) Combination of interval-valued fuzzy set and soft set [Computers & Mathematics with Applications 58: 521-527]. *Computers & Mathematics with Applications* 61: 1468-1470.
42. Liu X, Feng F, Jun YB (2012) A note on generalized soft equal relations. *Computers & Mathematics with Applications* 64: 572-578.
43. Feng F, Li Y (2013) Soft subsets and soft product operations. *Information Sciences* 232: 44–57.
44. Abbas M, Ali B, Romaguera S (2014) On generalized soft equality and soft lattice structure. *Filomat* 28: 1191–1203.
45. Abbas M, Ali M, Romaguera S (2017) Generalized operations in soft set theory via relaxed conditions on parameters. *Filomat* 31: 5955-5964.
46. Al Shami T (2019) Investigation and corrigendum to some results related to g soft equality and gf soft equality relations. *Filomat* 33: 3375-3383.
47. Al Shami TM, El Shafei ME (2020) T soft equality relation. *Turkish Journal of Mathematics* 44: 1427–1441.
48. Çağman N, Enginoğlu S (2010) Soft set theory and uni-int decision making. *European Journal of Operational Research* 207: 848–855.
49. Sezer AS (2012) A new view to ring theory via soft union rings, ideals and bi-ideals. *Knowledge-Based Systems* 36: 300–314.
50. Kaygisiz K (2012) On soft integral groups. *Annals of Fuzzy Mathematics and Informatics* 4: 363–375.
51. Sezer AS, Çağman N, Atagün AO, Ali MI, Türkmen E (2015) Soft intersection semigroups, ideals and bi ideals; a new application on semigroup theory I. *Filomat* 29: 917–946.
52. Sezgin A, Durak İ, Ay Z (2025) Some new classifications of soft subsets and soft equalities with soft symmetric difference–difference product of groups. *Amesias* 6: 16-32.
53. Sezgin A, Ay Z (2025) Soft intersection-difference product of groups. *Uncertainty Discourse and Applications* 2: 45-60.
54. Sezgin A, Aybek FN, Atagün AO (2023) A new soft set operation: Complementary soft binary piecewise intersection operation. *Black Sea Journal of Engineering and Science* 6: 330-346.
55. Sezgin A, Aybek FN, Güngör NB (2023) A new soft set operation: Complementary soft binary piecewise union operation. *Acta Informatica Malaysia* 7: 38–53.
56. Sezgin A, Çağman N (2024) A new soft set operation: Complementary soft binary piecewise difference operation. *Osmaniye Korkut Ata University Journal of the Institute of Science and Technology* 7: 1-37.
57. Sezgin A, Çağman N (2025) An extensive study on restricted and extended symmetric difference operations of soft sets. *Utilitas Mathematica*.
58. Sezgin A, Çağman N, Atagün AO (2023) Complemental binary operations of sets and their application to group theory. *Matrix Science Mathematic* 7: 114-121.
59. Sezgin A, Demirci AM (2023) A new soft set operation: Complementary soft binary piecewise star operation. *Ikonion Journal of Mathematics* 5: 24–52.
60. Sezgin A, Sarılioğlu M (2024) A new soft set operation: Complementary soft binary piecewise theta operation. *Journal of Kadirli Faculty of Applied Sciences* 4: 325–357.
61. Sezgin A, Sarılioğlu M (2024) Complementary extended gamma operation: A new soft set operation. *Natural and Applied Sciences Journal* 7: 15–44.
62. Sezgin A, Shahzad A, Mehmood A (2019) A new operation on soft sets: Extended difference of soft sets. *Journal of New Theory* 27: 33–42.
63. Sezgin A, Şenyiğit E (2025) A new product for soft sets with its decision-making: soft star-product. *Big Data and Computing Visions* 1: 52–73.
64. Sezgin A, Yavuz E (2023) A new soft set operation: soft binary piecewise symmetric difference operation. *Necmettin Erbakan University Journal of Science and Engineering* 5: 150–168.
65. Sezgin A, Yavuz E (2024) A New Soft Set Operation: Complementary soft binary piecewise lambda operation. *Sinop Üniversitesi Fen Bilimleri Dergisi* 8: 101–133.
66. Sezgin A, Yavuz E (2024) Soft binary piecewise plus operation: A new type of operation for soft sets. *Uncertainty Discourse and Applications* 1: 79–100.
67. Sezgin A, Aybek FN, Stojanovic N (2024) An in-depth analysis of restricted and extended lambda operations for soft sets. *Optimality* 1: 232–261.
68. Sezgin A, Yavuz E, Özlü Ş (2024) Insight into soft binary piecewise lambda operation: a new operation for soft sets. *Journal of Umm Al-Qura University for Applied Sciences* 10.
69. Sezer AS, Atagün AO, Çağman N (2013) A new view to N group theory: Soft N groups. *Fasciculi Mathematici* 51: 123–140.
70. Sezer AS, Atagün AO (2014) A new kind of vector space: Soft vector space. *Southeast Asian Bulletin of Mathematics* 40: 753–770.
71. Sezer AS, Atagün AO, Çağman N (2014) N group SI action and its applications to N group theory. *Fasciculi Mathematici* 52: 139–153.
72. Atagün AO, Sezer AS (2015) Soft sets, soft semimodules and soft substructures of semimodules. *Mathematical Sciences Letters* 4: 235–242.
73. Atagün AO, Sezgin A (2018) A new view to near ring theory: Soft near rings. *South East Asian Journal of Mathematics & Mathematical Sciences* 14: 1-14.

74. Atagün AO, Sezgin A (2015) Soft subnear rings, soft ideals and soft N subgroups of near rings. *Mathematical Sciences Letters* 7: 37–42.
75. Manikantan T, Ramasany P, Sezgin A (2023) Soft quasi ideals of soft near rings. *Sigma Journal of Engineering and Natural Science* 41: 565–574.
76. Atagün AO, Sezgin A (2017) Int soft substructures of groups and semirings with applications. *Applied Mathematics & Information Sciences* 11: 105–113.
77. Atagün AO, Sezgin A (2022) More on prime, maximal and principal soft ideals of soft rings. *New Mathematics and Natural Computation* 18: 195–207.
78. Tunçay M, Sezgin A (2016) Soft union ring and its applications to ring theory. *International Journal of Computer Applications* 151: 7-13.
79. Riaz M, Hashmi MR, Karaaslan F, Sezgin A, Shamiri MMAA, et al. (2023) Emerging trends in social networking systems and generation gap with neutrosophic crisp soft mapping. *CMES Computer Modeling in Engineering and Sciences* 136: 1759–1783.
80. Gulistan M, Feng F, Khan M, Sezgin A (2018) Characterizations of right weakly regular semigroups in terms of generalized cubic soft sets. *Mathematics* 6: 293.
81. Sezgin A, Orbay M (2022) Analysis of semigroups with soft intersection ideals. *Acta Universitatis Sapientiae Mathematica* 14: 166–210.
82. Sezgin A, İlgin A (2024) Soft intersection almost subsemigroups of semigroups. *International Journal of Mathematics and Physics* 15: 13–20.
83. Khan A, Izhar M, Sezgin A (2017) Characterizations of Abel Grassmann's groupoids by the properties of their double framed soft ideals. *International Journal of Analysis and Applications* 15: 62–74.
84. Sezgin A, Atagün AO, Çağman N, Demir H (2022) On near rings with soft union ideals and applications. *New Mathematics and Natural Computation* 18: 495–511.
85. Jana C, Pal M, Karaaslan F, Sezgin A (2019) (α, β) -soft intersectional rings and ideals with their applications. *New Mathematics and Natural Computation* 15: 333–350.
86. Sezgin A, Onur B (2024) Soft intersection almost bi ideals of semigroups. *Systemic Analytics* 2: 95–105.
87. Mahmood T, Rehman ZU, Sezgin A (2018) Lattice ordered soft near rings. *Korean Journal of Mathematics* 26: 503–517.
88. Sezgin A, Onur B, İlgin A (2024) Soft intersection almost tri ideals of semigroups. *SciNexuses* 1: 126–138.
89. Sezgin A, Aybek FN (2023) A new soft set operation: Complementary soft binary piecewise gamma operation. *Matrix Science Mathematic* 7: 27-45.
90. Aktas H, Çağman N (2007) Soft sets and soft groups. *Information Science* 177: 2726-2735.
91. Feng F, Jun YB, Zhao X (2008) Soft semirings. *Computers and Mathematics with Applications* 56: 2621-2628.
92. Ali MI, Mahmood M, Rehman MU, Aslam MF (2015) On lattice ordered soft sets. *Applied Soft Computing* 36: 499-505.
93. Atagün AO, Kamacı H, Taştekin İ, Sezgin A (2019) P-properties in near-rings. *Journal of Mathematical and Fundamental Sciences* 51: 152-167.
94. Ali B, Saleem N, Sundus N, Khaleeq S, Saeed M et al. (2022) A contribution to the theory of soft sets via generalized relaxed operations. *Mathematics* 10: 26-36.
95. Gulistan M, Shahzad M (2014) On soft KU-algebras. *Journal of Algebra, Number Theory: Advances and Applications* 11: 1-20.
96. Sezer AS, Çağman N, Atagün AO (2014) Soft intersection interior ideals, quasi-ideals and generalized bi-ideals; A new approach to semigroup theory II. *J. Multiple-Valued Logic and Soft Computing* 23: 161-207.
97. Ullah A, Karaaslan F, Ahmad I (2018) Soft uni-abel-grassmann's groups. *European Journal of Pure and Applied Mathematics* 11: 517-536.
98. Karaaslan F (2019) Some properties of AG^* -groupoids and AG -bands under SI -product Operation. *Journal of Intelligent and Fuzzy Systems* 36: 231-239.
99. Khan M, Ilyas F, Gulistan M, Anis S (2015) A study of soft AG -groupoids. *Annals of Fuzzy Mathematics and Informatics* 9: 621–638.
100. Mahmood T, Waqas A, Rana MA (2015) Soft intersectional ideals in ternary semiring. *Science International* 27: 3929-3934.
101. Memiş S (2022) Another view on picture fuzzy soft sets and their product operations with soft decision-making. *Journal of New Theory* 38: 1-13.
102. Sezgin A, Çağman N, Çıtak F (2019) α -inclusions applied to group theory via soft set and logic. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics* 68: 334-352.
103. Alcántud JCR, Khameneh AZ, Santos-García G, Akram M (2024) A systematic literature review of soft set theory. *Neural Computing and Applications* 36: 8951–8975.
104. Özlü Ş, Sezgin A (2020) Soft covered ideals in semigroups. *Acta Universitatis Sapientiae Mathematica* 12: 317-346.
105. Sezgin A, Onur B (2024) Soft intersection almost bi-ideals of semigroups. *Systemic Analytics* 2: 95-105.
106. Sezgin A, İlgin A (2024) Soft intersection almost ideals of semigroups. *Journal of Innovative Engineering and Natural Science* 4: 466-481.
107. Sezgin A, Durak İ, Ay Z (2025) Soft symmetric difference complement with soft intersection-symmetric difference complement product of groups. *Middle East Journal Of Science* 11.