Scientific Knowledge Group

ISSN: 3068-2789

Research Article

Journal of All Physics Research and Applications

Wave-focusing Materials

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ABSTRACT
Let $f \in L^2(S^2)$ be arbitrary, $\epsilon > 0$ be arbitrary small fixed, $q(x) \in L^2(D)$ be a real-valued function, $D \subset R^3$ be a bounded domain, $\alpha \in S^2$ be a fixed vector, $k > 0$ be a fixed number. Denote by $A(\beta)$ the scattering amplitude $A(\beta, \alpha, k)$, corresponding to the scattering of the plane wave $e^{ik\alpha\cdot x}$ by the potential q. The inverse problem is:

IP: Can one find q such that $||f(\beta) - A(\beta)|| < \epsilon$.

We solve this problem and give a method for its solution.

Keywords: Wave Scattering by Many Small Bodies, Smart Materials, Materials with a Desired Radiation Pattern.

Received: March 15, 2025;
Accepted: March 22, 2025;
Published: March 31, 2025

MSC: 26D10; 34G20; 7L05;44J05; 47J35;

Introduction

Let $f \in L^2(S^2)$ be arbitrary, $\epsilon > 0$ be arbitrary small fixed, $q(x) \in L^2(D)$ be a real-valued function, $D \subset R^3$ be a bounded domain, $\alpha \in S^2$ be a fixed vector, S^2 is the unit sphere in R^3, $k > 0$ be a fixed number. Denote by $A(\beta)$ the scattering amplitude $A(\beta, \alpha, k)$, corresponding to the scattering of the plane wave $e^{ik\alpha\cdot x}$ by the potential q.

The inverse problem is:

IP: Can one find q such that

$$||f(\beta) - A(\beta)|| < \epsilon, \qquad (1)$$

where $\epsilon > 0$ is an arbitrary small number?

It was not known if this problem has a solution. This problem was studied and solved in [3]. We formulate the result and refer the reader to monograph for a detailed proof [3]. The purpose of this paper is to give a detailed proof of a simpler result.

The scattering problem is:

$$\Delta u + k^2 u - q(x)u = 0, \quad u = u_0 + v, \quad u_0 = e^{ik\alpha\cdot x}, \quad \lim_{r\to 0} r(v_r - ikv) = 0, \quad (2)$$

where $r = |x|$. Problem (2) has a solution and the solution is unique (see, for example, [4]). The scattering amplitude is ([4], p. 363):

$$A(\beta) := A_q(\beta) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} e^{-ik\beta\cdot y} q(y,\alpha,k)u(y)dy, \qquad (3)$$

where $u(y) := u(y, \alpha, k)$ is the scattering solution, and the dependence on α and k is omitted since α and k are fixed. Since $\alpha \in S^2$ and $k > 0$ are fixed, we denoted the scattering amplitude $A(\beta)$.

Let us formulate our result.

Theorem 1. For any $f(\beta) \in L^2(S^2)$ and an arbitrary small number $\epsilon > 0$, there exists a $q \in L^2(D)$ such that inequality (1) holds.

Remark 1. There are infinitely many potentials satisfying (1). Indeed, the scattering amplitude depends continuously on the potential in the following sense:

$$\|A_{q_1} - A_{q_2}\|_{L^2(S^2)} \le c\|q_1 - q_2\|_{L^2(D)}, \qquad (4)$$

where $c > 0$ is a constant depending only on the bound for the norms of the potentials and on D. Therefore, small changes of the potential in $L^2(D)$ norm led to small changes in the scattering amplitude in $L^2(S^2)$ norm in the sense (4). Thus, if inequality (1) holds for some $q \in L(D)$, it will hold for any potential, sufficiently close to q in $L^2(D)$ norm. Estimate (4) can be obtained from equation (3) by using the Cauchy inequality and the estimate sup_{$x \in R^3$} $|u(x, \alpha, k)| < c$, ([4], p.364.)

Remark 2. Theorem 1 can be of practical interest. For example, let $f(\beta) = 1$ in

Citation: Alexander G Ramm (2025) Wave-focusing Materials. J All Phy Res Appli 1: 1-2.

a narrow cone and $f(\beta) = 0$ outside this cone. Then the body D with such radiation pattern will have practical interest. The wave, scattered by this body, will be scattered mostly in the above cone. The scattered wave can be directed not as usual to the back of the body and to the front of the body, but mostly to the above cone. Such bodies we call wave-focusing bodies.

Proofs

Let us prove first the following result.

Theorem 1a. For any $f \in L^2(S^2)$ and any $\epsilon > 0$ one can find $h \in L^2(D)$ such that the set $\int_D e^{-ik\beta\cdot x}h(x)dx\} \forall h \in L^2(D)$ is dense in $L^2(S^2)$.

Corollary 1. Given $f \in L^2(S^2)$ and $\epsilon > 0$, arbitrarily small, one can find $h \in L^2(D)$ such that

$$\|f(\beta) + \frac{1}{4\pi}\int_D e^{-ik\beta\cdot x}h(x)dx\| < \epsilon.$$

Proof of Theorem 1a. Assume the contrary. Then there exists $\psi \in L^2(S^2)$ such that

$$0 = \int_{S^2} d\beta\psi(\beta)\int_D e^{-ik\beta\cdot x}h(x)dx \quad \forall h \in L^2(D).$$

Changing the order of integration, one gets:

$$\int_{S^2} d\beta\psi(\beta)e^{-ik\beta\cdot x} = 0 \quad \forall x \in D \subset \mathbb{R}^3.$$

Since the integral is an entire function of x which vanishes in D, it vanishes in R^3 by analytic continuation. Therefore,

$$\int_0^\infty d\lambda\lambda^2\int_{S^2} d\beta e^{-i\lambda\beta\cdot x}\psi(\beta)\frac{\delta(\lambda-k)}{k^2} = 0 \quad \forall x \in \mathbb{R}^3.$$

By the injectivity of the Fourier transform, one gets

$$\psi(\beta)\frac{\delta(\lambda-k)}{k^2} = 0.$$

Applying this distribution to an arbitrary $C_0^\infty(\mathbb{R}^3)$ test function $\phi(\lambda, \beta)$ one gets $\psi(\beta)\phi(k, \beta) = 0$. Since the function ϕ is arbitrary, it follows that $\psi(\beta) = 0$.

Theorem 1a is proved.

Theorem 1 follows from Theorem 1a and the following Theorem 1b:

Theorem 1b. The set $\{q(x)u(x,\alpha,k)\}\forall q \in L^2(D)$ **is dense in** $L^2(D)$.

A proof of Theorem 1b is more difficult than the proof of Theorem 1a. It can be found in [3], pp. 61-65. Let us outline some of the ideas of our proof of Theorem 1b.

If the function

$$q(x) = \frac{h(x)}{u(x)}, \quad u = u(x) = u(x,\alpha,k) \qquad (5)$$

belongs to $L^2(D)$, then $qu \in L^2(D)$, and there is nothing to prove. If $q(x)$ does not belong to $L^2(D)$, then $u = 0$, so $\Re u = 0$ and $\Im u = 0$. These are two equations for the H^2 functions in R^3, where $H^2 = H^2(R^3)$ is the Sobolev space. Therefore, they define a line in R^3. Let us cover this line by a tubular neighborhood $N_\delta := \{x : x \in D, |u(x)| \le \delta\}$ and let $D_\delta := D \setminus N_\delta$. Define $h_\delta = h$ in D_δ and $h_\delta = 0$ in N_δ, and set $q_\delta := \frac{h_\delta}{u_\delta}$, where $u_\delta = u_0 - \int_D g(x,y)h_\delta(y)dy$.

One can prove that this q_δ is a bounded function [3]. This proves Theorem 1b.

Conclusion

A recipe is given for creating materials with a desired refraction coefficient by embedding into a given material many small particles with prescribed boundary impedances [3].

The refraction coefficient can be so chosen that the resulting material will have a desired radiation pattern for a fixed wave number and a fixed direction of the incident plane wave.

Materials with a prescribed radiation pattern (wave-focusing materials) can be created.

For future developments it is desirable to do many experiments, based on the author's theory. One can change the given refraction coefficient $n_0(x)$ in a desired direction.

Theoretically, the major advance is the author's (asymptotical as $a \to 0$) solution to many-body scattering problem under the assumption $a \ll d \ll \lambda$.

Disclosure Statement

There are no competing interests to declare. There is no financial support for this work.

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